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Technical Report

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Smooth Sliding Mode Controller Design for Robust Missile Autopilot

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Smooth Sliding Mode Controller Design for Robust Missile Autopilot

Description of the Effort

a. General

The University of Alabama in Huntsville (UAH) will provide the personnel and equipment to develop Smooth Sliding Mode Control algorithms. These algorithms may be used to provide a generic flight control to advanced guided weapon systems.

b. Requirements

The developed Sliding Mode Controller (SMC) algorithms will be robust to agile reference commands and unknown disturbances from environmental conditions (atmospheric turbulence), as well as missile and its actuator uncertainties and failures. The key task in developing a smooth SMC is to design a multi-loop control system, where the sliding mode exists in every loop. A virtual control signal to be produced in every outer loop must be smooth, for it has to be tracked by inner cascades of a multi-loop system. Smoothness of a control signal and existence of a multi-loop SMC must be achieved without degrading high precision and robustness of sliding modes. The developed smooth sliding mode control algorithms are to be applied to generic missile control problems.

Abstract

Presented is a method of smooth sliding mode control design to provide for the second-order sliding mode on the selected sliding surface. The control law is a nonlinear dynamic feedback that in absence of unknown disturbances provides for finite-time convergence of the second-order reaching phase dynamics. The application of the second-order disturbance observer in a combination with the proposed continuous control law gives the second-order sliding accuracy in presence of unknown disturbances and the discrete-time control update. The piecewise constant control feedback is smooth.

The proposed control scheme is applied to a generic three-loop control system, and the proposed multiple time scale sliding mode design is demonstrated on the simplified model of a ballistic interceptor missile.

Contents

1. Introduction

The area of advanced missile guidance and control is a promising field of application for latest developments in robust control theory. Many generations of highly successful missiles use different variants of classical algorithms such as proportional navigation guidance and PID control [1-3] in association with various extended Kalman filtering techniques. This is the major reason for industry designers to continue advancement and modernization of these relatively simple topologies and linear autopilots with fixed given structure. However, "emerging threats and new operational constraints have created the need for a variety of new weapon systems, .., even existing systems can expect significant upgrades."[4] The demand for engaging the highly agile targets [5] with hit-to-kill accuracy and resistance to electronic countermeasures, combined with that for increased standoff range or high off-boresight angle and effective reconfiguration for different missions, create new challenges for future missile design. There one can observe three main directions of rapid theory and technology proliferation to address the need:

- advanced navigation equipment and estimation algorithms;
- advanced guidance strategies;
- advanced missile airframes.

In the last direction, the high performance vehicles are designed to be open-loop unstable, to be non-axisymmetric for extreme agility; new control actuating mechanisms, including hot- or cold-gas thrusters and thrust vectoring, hybrid rocket motors, reduced surface area high-speed fins etc., are in development to increase maneuverability over expanded envelope of flight conditions.

In many cases, integration of new guidance strategy with new airframe (hybrid rocket motor with LQG optimal guidance law [6], Sliding Mode Control based guidance with pulse type on-off thrusters [7]) directly accounts for increased performance in missile airframe due to the mentioned above developments only, leaving the autopilot structure to be the most conservative area for design upgrades, and frequently reducing the tracking control problem to control allocation only [6].

Missile autopilots are usually designed to track acceleration issued by a guidance law on the basis of data sensed and estimated by a navigation system, where a widely accepted approach is to estimate relative position, velocity and acceleration via extended Kalman filtering technique. The typical autopilot uses the gain scheduling technique or rather different parameter

optimization methods (see a review in [4]), given a particular fixed controller structure, such as a classical PID architecture. The simplest control structure is easy to implement, to tune up and validate for different performance specifications. The industry designers prefer it to recent linear and nonlinear schemes, because they never really know how these novel techniques of high complexity or high-order dynamics would trade off performance for uncertainty robustness, or improve them both significantly, or whether they will work at all. Moreover, many design approaches pursue the idea that the problem to design a good controller is to obtain a good mathematical model of the plant, to derive sophisticated equations, which can include as much known effects as possible and utilize as much as possible the data obtained via hardware-in-the-loop simulations.

Having said that, one can come to the following conclusion.

1.1 Requirements for Advanced Missile Autopilot

A new missile autopilot of practical interest should have

- A simple clear structure, which can be readily designed, facilitated, implemented, tuned up and monitored;
- Increased both performance and robustness to visible level;
- Required the least possible modeling information to guarantee these performance and robustness;
- Relaxed the sensor/processor/actuator requirements without introducing new restrictive assumptions, which is difficult to verify;
- Maximal autonomy with respect to common sensed data to be easily integrated with different navigation/guidance systems;
- Required all on-the-fly ad hoc inevitable modifications being conducted by a professional with common relevant control theory background.

Having almost all the mentioned above characteristics, the Sliding Mode Control (SMC) systems [8-12] are valued in the class of robust, high fidelity control algorithms for their robust accommodation of uncertainties and cancellation of disturbances. In addition, a new generation of sliding mode algorithms (high order sliding modes [13],[14]) featuring usually 1st or 2nd order dynamics in traditionally static nonlinear state feedback (conventional sliding mode), produces continuous control input, which can explicitly account for not only actuator saturation but also rate saturation limit. The other additional features include enhanced robustness to measurement

noise and opportunity to realize output feedback instead of state feedback to increase robustness to parametric and dynamic uncertainties.

1.2 SMC as a Perspective Control Technique for Advanced Missile Autopilot

Further, we briefly outline the SMC design specifications to address some of the formulated requirements. A SMC system has a simple controller structure, clear distinct design steps and performance assessment. It starts with a highly uncertain plant model, where the robustness analysis (Lyapunov function technique) is included in controller design. The only known information required is the relative degree of each commanded output and the ranges of uncertainties for unknown parameters, nonlinear terms, external disturbances and measurement noise, as well as actuator limits and rate limits. Uncertain terms in certain subspace can be non-smooth. For noisy signals, the value of the second time derivative above which a signal can be considered as noise is required.

The first step in controller design is to judiciously select the sliding manifold (sliding surface) in the system state space. Equations of sliding surface and the internal dynamics of the plant compose the closed-loop (compensated) dynamics of the plant. In minimum phase case, equations of sliding manifold are the desired tracking error dynamics, which is uncoupled from the internal dynamics and completely plant independent. The system motion on this manifold (the sliding mode) will be absolutely robust to plant uncertainties and external disturbances. The order of asymptotic or finite-time-convergence tracking-error dynamics, which can be enforced in sliding mode is equal to r-1, where r is the relative degree of the input/output plant dynamics. Finite-time-convergence introduces enhanced robustness to perturbations due to the fact that convergence rate in the vicinity of the origin will be higher than that of asymptotic linear convergence.

The second step in the SMC design is to design a control law to provide convergence to and stability on the sliding surface $\sigma = 0$. The control law [8] usually consists of two additive terms

$$u = \hat{u}_{eq} + u_{un}\,. \tag{1.1}$$

In case of full information of the plant and external inputs, the term u_{eq}, called *equivalent control* [8], can be determined to provide exact keeping $\sigma = 0$. In uncertain case it's substituted by an estimate, $\hat{u}_{eq}$, of any portion of u_{eq}, which can be estimated (calculated). All the rest of

uncertainty is captured by another part, u_{un}, which is additive and complimentary to $\hat{u}_{eq}$ and can capture even totally uncertain u_{eq}, accepting $\hat{u}_{eq} = 0$.

For conventional sliding mode $\sigma = 0$, the examples of u_{un} are

- discontinuous control [8]

$$u_{un} = \rho \operatorname{sgn}(\sigma), \tag{1.2}$$

- continuous control [16]

$$u_{un} = \tilde{\rho} \cdot \sqrt[3]{\sigma}. \tag{1.3}$$

For the second-order sliding mode $\dot{\sigma} = \sigma = 0$ continuous control law [14] is

$$u_{un} = \rho_1 \frac{\sigma}{|\sigma|^{0.5}} + \rho_0 \int \operatorname{sgn}(\sigma) d\tau. \tag{1.4}$$

All the uncertainties of the plant are accumulated in a single quantity, the relative uncertainty of equivalent control $|u_{eq} - \hat{u}_{eq}| < L$, which should be bounded in a reasonable domain of its arguments. The relative uncertainty gives the value for coefficients $\rho, \tilde{\rho}, \rho_1, \rho_0$ to get the sufficient control authority to provide existence of sliding mode. The greater relative uncertainty, the more control authority is required to guarantee compensation for it. (The method to get rigorous proof to existence of sliding mode [8] is based on the Lyapunov function technique.)

1.3 SMC Implementation Issues

In practical implementation of the control law (1.2) or (1.4), due to finite time switching of $\operatorname{sgn}(\cdot)$, the very high frequency auto-oscillation regime is established. It has the ability to capture all the uncertainties and disturbances and cancel out their effect on the closed-loop dynamics. It has robustness to asynchronous sample rates of measurements and ZOH control in digital controllers as well. Control (1.2) provides for accuracy of $\sigma = 0$ to be proportional to τ (switching time of $\operatorname{sgn}(\cdot)$). The continuous control (1.4) provides for accuracy of $\sigma = 0$ to be proportional to τ^2. The simplest control law (discontinuous (1.2) or continuous (1.3)) provides for given robustness. All further sophistication (for control to operate at almost maximal amplitude and rate without saturation, or to provide the output feedback only, instead of state feedback) increase performance and retain robustness.

The discontinuous SMC is relevant for missile autopilot

There are two major objections against discontinuous control law: the on-off switchable mode is not energy saving and, in case when control is the fin's deflection angle, they cannot move instantly. However, discontinuous control law is particularly applicable to the pulse type on-off thrust rocket engines with pulse frequency, which is much higher than the bandwidth of the overall system dynamics, where the PWM (Pulse Width Modulation) or PWPF are employed so far [7] to produce input equivalent to the continuous control law. Besides, many amplifiers in electromechanical actuators work in PWM mode to transmit the control signal to the amplified actual power signal. Instead of PWM, a SMC will automatically produce high frequency auto-oscillation.

The control structure (1.1) leaves the place for any parallel adaptive corrections (based on explicit or implicit on-line identification). The less u_{eq} becomes uncertain, the better. The direct link of uncertainty bounds to control authority gives the room for further optimization of specifications on control effectors design. In case of severe failure or condition change, which is above the range specified initially, the automatic reconfiguration can be made [17] without change of main structure.

At last, one should mention that SMC design is less theory intensive than many novel approaches, such that a customer can operate independently with the new technology being delivered. Thus, the simple fixed controller structure for a plant with given relative degree and the given range for cumulative relative uncertainty, and clear distinct steps to design a controller and to tune it up, make SMC to be a promising technique for practical missile autopilot design.

In this report, we develop a multiple loop control system featuring sliding modes in each loop and demonstrate the application of this SMC design to a generic missile autopilot

2. Missile Autopilot General Structure

The first proposal of this work is to establish a new autopilot structure, and then to apply all the corrections, adaptations, gain scheduling, if necessary. For decades, a classical generalized linear autopilot structure to track the commands, issued by a guidance law, has had the form presented in Fig.1 [3]

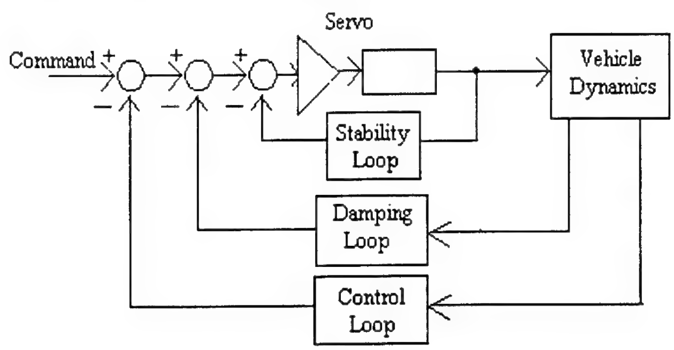

Fig.1 Classical Autopilot Structure

Where, the stability loop is to stabilize the servo in case it's a pure integrator; the damping loop is necessary to restrict oscillations and overshoots to provide adequate system stability margins; the control loop is responsible for command signal tracking.

In the missiles, the electrical control signal furnished by the controller is superimposed on a direct rate gyro feedback implementing damping loop. Sometimes, a linear feedback controller is accomplished by adding a lead compensator to the forward loop to stabilize the unstable airframe [2]. Many new linear and nonlinear design approaches have left the first two loop without change and redesign only the third, servo-tracking control loop [11]. As far as the SMC enforces the closed-loop dynamics with parameters of entirely designer's choice and the order being equal to the relative degree of input/output dynamics, this classical structure is no more necessary. Moreover, to provide the extreme agility of a missile, this artificial damping loop will be neutralized by a SMC and will not be effective anymore. The plant can even have an unstable actuator, since all the problems with stability and performance will be resolved by a SMC.

For instance, the following structure for attitude angles tracking is proposed for minimum phase plants.

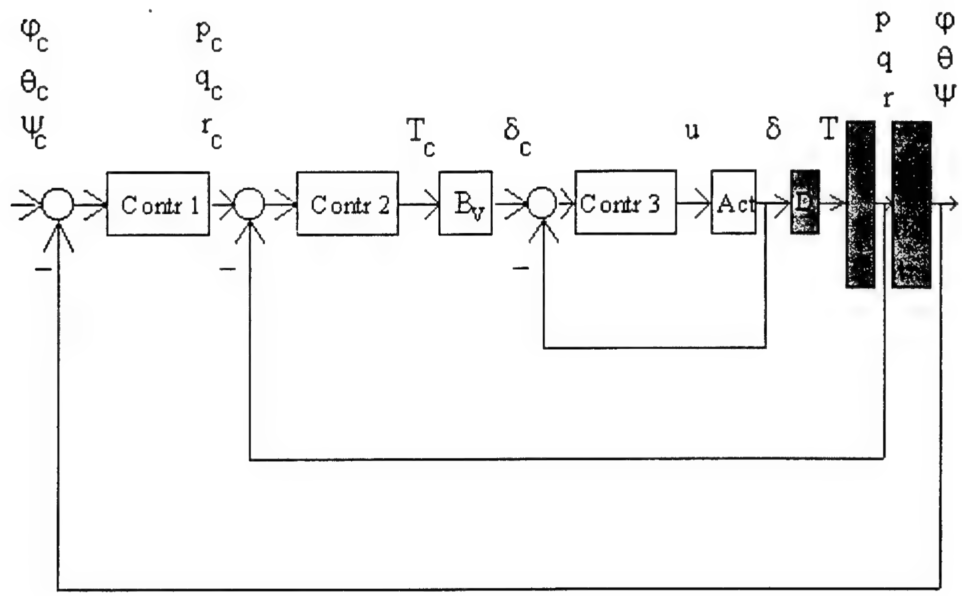

Fig.2 SMC based multiple loop missile autopilot (General Scheme)

where the sliding quantity, σ, is formed in each loop, and enforced via virtual control that is to be followed by in the next inner loop.

In the next section, we design a multiple loop SMC system.

3. Smooth Sliding Mode Control in Multiple Loop Tracking Systems

3.1 Introduction

In multiple-loop backstepping-type control systems, an important issue is to provide a so-called virtual control signal to be smooth in an outer loop, for it has to be tracked by inner cascades of a multi-loop system. Smoothness of a control signal is not easy achievable in control systems with sliding modes without degrading high precision and robustness.

The idea of this work is to use combination of both the sliding mode estimator for a plant disturbance and smooth sliding mode controller to design a smooth sliding mode controller that is robust to disturbances and provides finite time convergence to the custom made sliding surface.

Two types of smooth sliding mode controllers are to be designed.

- First order smooth finite reaching time sliding mode controller with a traditional sliding mode observer for disturbance observation. Under the discrete-time control with a zero-order hold, the accuracy of holding the trajectories on the sliding surface is of the first order real sliding $O(T)$.
- Second order smooth finite reaching time sliding mode controller with a second order sliding mode observer for disturbance observation. Under the discrete-time control with a zero-order hold, the accuracy of holding the trajectories on the sliding surface is of the second order real sliding $O(T^2)$.

For many control applications, Sliding Mode Control (SMC) [8-10] has been proven efficient technique to provide high-fidelity performance in different control problems for nonlinear systems with uncertainties in system parameters and external disturbances. Ideal sliding modes feature theoretically-infinite-frequency switching, while the real conventional sliding modes feature high finite frequency switching of an input signal (control). Such a mode might be unacceptable if the control signal has to be tracked by inner cascades of a multi-loop system. Trading the absolute robustness on the sliding surface for the system convergence to a small domain, the "boundary layer", around it under a continuous control law, the methods of this group employ a high-gain saturation function [8],[19] or [20] a sigmoid function. In the continuous-time control systems with sampled-data measurements and/or discrete-time control action (zero-order hold with digital control), different types of closed-loop boundary-layer

dynamics are employed to provide for a smooth control, varying from self-adaptive saturation level functions [21] to fixed-gain deadbeat controls with disturbance estimation using delayed-time data [22-24]. Another alternative [25] to the latter approach is to incorporate into the "boundary-layer" dynamics an exosystem model for disturbances [26] avoiding the direct observer-based disturbance estimation.

The idea to hide discontinuity of control in its higher derivatives has been realized using higher order sliding modes [14,27]. The resulting higher-order sliding mode is of enhanced accuracy and robustness to disturbances. However, a drawback of the direct application of this approach to chattering attenuation is that it cannot tolerate unmodeled fast dynamics. Therefore, the designed continuous control cannot be, for instance, an outer-loop feedback in a multi-loop control system.

The idea of this paper is to use both the disturbance estimation and the higher-order sliding mode techniques to design a continuous sliding mode control, providing finite-time convergence to the sliding surface and establish the second-order sliding mode in absence of unknown disturbances. In case when disturbances are present, the disturbance observer determines the accuracy. Employing the second-order observer [14], the second-order sliding accuracy can be achieved. The main contribution of this paper is in further development of the approach presented in the work [28].

3.2 Tracking Problem Formulation

Consider a MIMO plant with n states and m controls, where the "diagonalization method" [8] has been applied producing m independent dynamics for each input-output channel. Then, consider the following SISO nonlinear uncertain system that can represent any input-output channel (we assume relative degree is equal to 3, although the given approach can be generalized to r^{th} order system as well)

$$\begin{aligned}
\dot{x}_1 &= x_2 + \varphi_1(x_1) + f_1(x_1, t), \\
\dot{x}_2 &= x_3 + \varphi_2(x_1, x_2) + f_2(x_1, x_2, t), \\
\dot{x}_3 &= \varphi_3(x_1, x_2, x_3) + f_3(\cdot, t) + u, \\
y &= x_1
\end{aligned} \tag{3.1}$$

where the $\varphi_i(\cdot)$-functions are known, $f_i(\cdot, t)$ are uncertain time-varying functions that are bounded in any bounded compact set of their arguments. The problem is to provide for the

output tracking: $y(t) \to y^c(t)$. The **design problem** is to achieve this tracking in multiple loop sliding modes using full state feedback and backstepping design.

3.3 Multiple Sliding Surface Design with 1st-Order Sliding Modes

A multiple sliding surface implementation of the back-stepping approach looks as follows [30].

1) Define the first sliding surface

$$S_1 = y^c - y = x_1^c - x_1 = 0, \tag{3.2}$$

and the following desired closed-loop dynamics for the sliding quantity S_1

$$\dot{S}_1 = -\gamma_1(S_1) + S_2 + \tilde{e}_1 \tag{3.3}$$

where $S_2 = x_2^c - x_2$, $x_2^c(t)$ is to be defined, $\tilde{e}_1 = \left(\dot{x}_1^c - f_1(\cdot,t)\right) - \left(\hat{\dot{x}}_1^c - \hat{f}_1\right)$, $x_1^c = y^c$, and $\gamma_1(\cdot)$ is such a function that the homogeneous part of the system (3.3) is a finite time convergent equation.

From (3.1)-(3.3) one can formally obtain

$$\dot{S}_1 = \dot{y}^c - x_2 - \varphi_1(\cdot) - f_1(\cdot,t) = -\gamma_1(S_1) + x_2^c - x_2 + \left(\dot{x}_1^c - f_1(\cdot,t)\right) - \left(\hat{\dot{x}}_1^c - \hat{f}_1\right) \tag{3.4}$$

From (3.4) we derive the following reference command for the next step

$$x_2^c = \hat{\dot{x}}_1^c - \hat{f}_1 - \varphi_1(x_1) + \gamma_1(S_1) \tag{3.5}$$

where $\hat{\dot{x}}_1^c, \hat{f}_1$ are the best estimates of the reference signal and uncertainty in the first loop that are based on dynamic observers.

2) The second sliding surface is defined as

$$S_2 = x_2^c - x_2 = 0, \tag{3.6}$$

Assuming existence of (3.6) and finite time convergence of (3.3) even with $\tilde{e}_1 \neq 0$, the sliding mode on the surface (3.2) will be achieved.

Sliding quantity S_2 is calculated using feedback on x_2 and equation (3.5) for x_2^c. An alternative way to produce S_2 using S_1 feedback only can be obtained as follows. Derive the following identity from (3.1), (3.2)

$$\dot{x}_1^c - f_1 = \dot{S}_1 + x_2 + \varphi_1(x_1),$$

then one can estimate

$$\hat{\dot{x}}_1^c - \hat{f}_1 = \hat{\dot{S}}_1 + x_2 + \varphi_1(x_1). \tag{3.7}$$

From (3.5),(3.7) we obtain

$$S_2 = \hat{\dot{S}}_1 + \gamma_1(S_1). \tag{3.8}$$

There are many ways to obtain (3.8). One can estimate the derivative and calculate the nonlinear term, one can try to estimate them both at once; at last, one can apply the nonlinear DSM approach and enforce (3.8) in the system motion in an auxiliary DSM.

The closed-loop dynamics for the sliding quantity S_2 is selected in the form

$$\dot{S}_2 = -\gamma_2(S_2) + S_3 + \tilde{e}_2 \tag{3.9}$$

where $S_3 = x_3^c - x_3$, $x_3^c(t)$ is to be defined, $\tilde{e}_2 = \left(\dot{x}_2^c - f_2(\cdot,t)\right) - \left(\hat{\dot{x}}_2^c - \hat{f}_2\right)$, and $\gamma_2(\cdot)$ is of the same class as $\gamma_1(\cdot)$. Similar to Eq.(3.5) the reference command $x_3^c(t)$ is obtained

$$x_3^c = \hat{\dot{x}}_2^c - \hat{f}_2 - \varphi_2(x_1, x_2) + \gamma_2(S_2)\,. \tag{3.10}$$

3) Finally, the third sliding surface,

$$S_3 = x_3^c - x_3 = 0\,, \tag{3.11}$$

is achieved under the control

$$u = -\varphi_3(\cdot) + k_{1,3}S_3 + k_{0,3}\,\mathrm{sgn}(S_3)\,. \tag{3.12}$$

The closed-loop system in the (S_1, S_2, S_3) state space is derived as

$$\begin{aligned}
\dot{S}_1 &= -\gamma_1(S_1) + S_2 + \tilde{e}_1, \\
\dot{S}_2 &= -\gamma_2(S_2) + S_3 + \tilde{e}_2, \\
\dot{S}_3 &= \dot{x}_3^c - f_3(\cdot,t) - k_{1,3}S_3 - k_{0,3}\,\mathrm{sgn}(S_3).
\end{aligned} \tag{3.13}$$

Dynamics (3.13) plus the error dynamics in the dynamic observers that govern $(\tilde{e}_1, \tilde{e}_2)$ complete the total closed-loop dynamics of the plant.

In the work [30], it's proved the closed-loop stability and convergence of (3.13) with first-order linear dynamic observers. These observers have served as integral filters to smooth out possibly non-Lipshitz behavior of $\dot{x}_1^c, \dot{x}_2^c, \dot{x}_3^c$. That's why it was possible to define feedback terms in each compensated dynamics as $\gamma_i(S_i) = k_{i,1}S_i + k_{i,0}\,\mathrm{sgn}(S_i), i = \overline{1,3}$. These functions provide for finite time convergence in (3.13), but they are not differentiable. Linear smoothing filters of the work [30] overcome the problem of "explosion of terms" trading robustness for stability. However, in numerical implementations this discontinuous form is rarely used, for even with filtering the term $\mathrm{sgn}(S)$ behavior in outer loops is not good for inner tracking loops. There is no finite time convergence and the true sliding mode [20] in this case. Another approach is developed in this work to establish finite time convergence in each loop to a dynamic integral-type sliding surface.

Additionally, the integral part specifies time scale separation [17] between the loops to ensure overall stability of a backstepping-type tracking system.

3.4 Multiple Dynamic Sliding Surface Design with 2nd-Order Sliding Modes

3.4.1 Continuous Control for Finite Reaching Time Sliding Mode

Finite-reaching-time continuous standard-sliding-mode controllers have been developed in works [16],[28] for the first order σ-dynamics

$$\dot{\sigma} = f(\sigma,t) + u \,. \tag{3.14}$$

They provide for finite-time-convergence of the first-order closed-loop σ-dynamics. One of the forms in the work [16] is given by

$$\dot{\sigma} + \rho \frac{\sigma}{|\sigma|^{0.5}} = 0 \,, \tag{3.15}$$

see Fig.3.

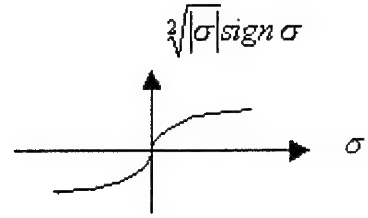

Fig.3 Nonlinear Terminal Sliding Manifold

In absence of uncertainty in the function $f(\sigma,t)$, the control law

$$u(\sigma) = -f(\sigma,t) - \rho \frac{\sigma}{|\sigma|^{0.5}} \,, \tag{3.16}$$

renders the closed-loop dynamics (3.15), as a finite time convergent nonlinear manifold. When the function $f(\sigma,t)$ is totally uncertain, the continuous control law

$$u(\sigma) = -\rho \frac{\sigma}{|\sigma|^{0.5}} \,, \tag{3.17}$$

provides for convergence to the arbitrarily small domain of attraction, the boundary layer, around the sliding surface $\sigma = 0$ in a standard sliding mode, where the gain ρ and the uncertainty limit L determine the boundary layer thickness. The drawbacks of this controller are that the uncertainty limit defines the boundary layer, and even in absence of uncertainty the domain of attraction to $\sigma = 0$ is proportional to the discrete interval T under the discrete-time control (first-

order sliding accuracy). An additional problem for a mupltiple loop system is that (3.17) is not smooth enough to be r times differentiable.

3.4.2 Conditions on Smoothness of a Virtual Control

In order for the control of form (3.17) to be a virtual control to be followed by inner cascades of total order r of a multi-loop system, it has to be r times continuously differentiable. In work [16], finite time convergence has been proved for the closed-loop dynamics

$$\dot{\sigma} + \rho|\sigma|^{\alpha}\,\mathrm{sgn}(\sigma) = 0\,, \tag{3.18}$$

where $\alpha \in (0,1)$. Similar to (3.17), it gives us the virtual control in the form

$$u_{(1)} = -\rho|\sigma|^{\alpha}\,\mathrm{sgn}(\sigma)\,. \tag{3.19}$$

For $u_{(1)}$ to be followed by a first order tracking system is has to be one time continuously differentiable. From (3.18),(3.19) we have

$$\dot{u}_{(1)} = -\rho^{2}\alpha|\sigma|^{2\alpha-1}\,\mathrm{sgn}(\sigma)\,. \tag{3.20}$$

Eq. (3.20) gives the following condition on smoothness of the virtual control at the origin

$$\alpha > \tfrac{1}{2}\,.$$

In case of an r^{th} order tracking system, we have

$$u^{(r)}{}_{(r)} \sim -|\sigma|^{(r+1)\alpha-r}\,\mathrm{sgn}(\sigma)\,. \tag{3.21}$$

Finally, combining conditions of finite time convergence of the compensated dynamics and smoothness of the virtual control, we obtain the following condition for the r^{th} order tracking system.

Condition on Smoothness. Terminal sliding mode can be enforced on the surface (3.18) by an r^{th} order tracking system in all loops if

$$\frac{r}{r+1} < \alpha < 1\,. \tag{3.22}$$

Applying this condition to system (3.1) we must have $\tfrac{2}{3} < \alpha < 1$ if we select $\gamma_1(S_1) = |S_1|^{\alpha}\,\mathrm{sgn}(S_1)$ for (3.3). One should not worry about overall stability of system (3.13) and convergence rate of the estimation error $(\widetilde{e}_1, \widetilde{e}_2)$-dynamics if time scale is applied in each loop. Multiple time scale is achieved in dynamic sliding surfaces designed in the next section. Second order sliding modes are established on dynamic sliding surfaces that govern compensated time scaled dynamics in each loop.

3.4.3 Multiple Time Scale Dynamic Sliding Surfaces

The first (outer) loop design

Consider the tracking problem for system (3.1). We introduce the following dynamic sliding surface for the compensated dynamics in the outer loop of the expected 3-loop system

$$\sigma_1 + \rho_1 \int |\sigma_1|^{\beta_{1,1}} \operatorname{sgn}(\sigma_1)dt = e_1 + c_1 \int |e_1|^{\alpha_1} \operatorname{sgn}(e_1)dt\,, \tag{3.23}$$

where e_1 is the input and the dynamic sliding surface quantity σ_1 is the output. If the sliding mode exists on the surface $\sigma_1 = 0$, then from (3.23) we obtain in sliding mode

$$e_1 + c_1 \int |e_1|^{\alpha_1} \operatorname{sgn}(e_1)dt = const\,,\ \sigma_1 = 0\,,$$

or

$$\dot{e}_1 = -c_1 |e_1|^{\alpha_1} \operatorname{sgn}(e_1)\,, \tag{3.24}$$

which is a finite time convergent system, where the terminal time is a function of $e_1(0), \alpha_1, c_1$. To enforce convergence to $\sigma_1 = 0$, we consider σ_1-dynamics using Eq.(3.1) and introducing the virtual control x_2^c, and the tracking error $e_2 = x_2^c - x_2$ of the inner loop

$$\dot{\sigma}_1 + \rho_1 |\sigma_1|^{\beta_{1,1}} \operatorname{sgn}(\sigma_1) = \dot{y}_c - \varphi_1(x_1) - f_1(.,t) - (x_2^c - e_2) + c_1 |e_1|^{\alpha_1} \operatorname{sgn}(e_1)\,. \tag{3.25}$$

The virtual control x_2^c is designed as

$$x_2^c = \hat{\dot{y}}_c - \varphi_1(x_1) - \hat{f}_1(.,t) + c_1 |e_1|^{\alpha_1} \operatorname{sgn}(e_1) + \rho_0 \int |\sigma_1|^{\beta_{1,0}} \operatorname{sgn}(\sigma_1)dt\,, \tag{3.26}$$

Given (3.26) and estimation error $\hat{e}_1 = (\dot{y}_c - f_1(.,t)) - (\hat{\dot{y}}_c - \hat{f}_1(.,t))$, the outer loop σ_1-dynamics closed under control (3.26) is obtained

$$\dot{\sigma}_1 + \rho_1 |\sigma_1|^{\beta_{1,1}} \operatorname{sgn}(\sigma_1) + \rho_0 \int |\sigma_1|^{\beta_{1,0}} \operatorname{sgn}(\sigma_1)dt = \hat{e}_1 + e_2\,. \tag{3.27}$$

Assuming exact estimation, i.e., $\hat{e}_1 = 0$, and fast convergence of e_2 to zero (time scale of the inner loop dynamics), we have the second order σ_1-dynamics

$$\ddot{\sigma}_1 + \rho_1 \frac{\dot{\sigma}_1}{|\sigma_1|^{1-\beta_{1,1}} \operatorname{sgn}(\sigma_1)} + \rho_0 |\sigma_1|^{\beta_{1,0}} \operatorname{sgn}(\sigma_1) = 0\,, \tag{3.28}$$

which is a finite time convergent differential equation given conditions formulated in the following lemma.

Lemma 1 Consider the following σ-dynamics

$$\dot{\sigma} = -\alpha_1 |\sigma|^{1/2} \operatorname{sgn}(\sigma) - \alpha_0 \int |\sigma|^{1/3} \operatorname{sgn}(\sigma)d\tau\,, \tag{3.29}$$

$\alpha_1 > 0, \alpha_0 > 0$, which can be equivalently presented by the system of two first-order equations

$$
\begin{cases} \dot{x}_1 = x_2 - \alpha_1 |x_1|^{1/2} \operatorname{sgn}(x_1), \\ \dot{x}_2 = -\alpha_0 |x_1|^{1/3} \operatorname{sgn}(x_1), \end{cases} \tag{3.30}
$$

where $x_1 = \sigma$. The system (3.29) is asymptotically stable and, moreover, it approaches the origin in a finite time.

Proof: It's not difficult to prove asymptotic stability of system (3.29), let a Liapunov function candidate be

$$
V(x_1, x_2) = \frac{x_2^2}{2} + \int_0^{x_1} \alpha_0 |z|^{1/3} \operatorname{sgn}(z) dz,
$$

$V(x) > 0,$ if $x \in \Re^2 \setminus \{0\}$, then the Liapunov function derivative will be

$$
\dot{V} = \frac{\partial V}{\partial x} \cdot \begin{bmatrix} x_2 - \alpha_1 |x_1|^{1/2} \operatorname{sgn}(x_1) \\ -\alpha_0 |x_1|^{1/3} \operatorname{sgn}(x_1) \end{bmatrix} = -\alpha_0 \alpha_1 |x_1|^{\frac{1}{3}+\frac{1}{2}} < 0, \text{ if } x \in \Re^2 \setminus \{0\}.
$$

However, in order to prove the finite time convergence to the origin of system (3.29), one have to apply special topics of Liapunov analysis of the finite time convergence differential equations. In the work [29] the following theorem has been proved.

Theorem Let $x \in D \subset \Re^n$, $\dot{x} = f(x)$, $f : \Re^n \to \Re^n$ is continuous on an open neighborhood D of the origin and locally Lipshitz on $D \setminus \{0\}$ and $f(0) = 0$. Suppose there exists a continuous function $V : D \to \Re$ such that the following conditions hold

(i) V is positive definite;

(ii) $\dot{V}$ is negative on $D \setminus \{0\}$;

(iii) there exist real numbers $k > 0$ and $\alpha \in (0,1)$, and a neighborhood $N \subset D$ of the origin such that $\dot{V} + kV^{\alpha} \le 0$ on $N \setminus \{0\}$.

Then, the origin is a finite-time-stable equilibrium of $\dot{x} = f(x)$.

The function V is satisfying (i),(ii) of the theorem. To prove (iii) we consider the validity of

$$\dot{V} + kV^{\alpha} = -\alpha_0\alpha_1|x_1|^{\frac{5}{6}} + k\left(\frac{x_2^2}{2} + \tfrac{3}{4}\alpha_0|x_1|^{\frac{4}{3}}\right)^{\alpha} \le 0 .$$

The inequality can be transformed to

$$k\left(\frac{x_2^2}{2} + \tfrac{3}{4}\alpha_0|x_1|^{\frac{4}{3}}\right) \le (\alpha_0\alpha_1)^{\frac{1}{\alpha}}|x_1|^{\frac{5}{6}\frac{1}{\alpha}} . \tag{3.31}$$

In a small neighborhood of the origin, where $x_2 << 1, x_1 << 1$, and $\dot{x}_1 \to 0$, i.e. $x_2 - \alpha_1|x_1|^{1/2}\operatorname{sgn}(x_1) \to 0$, we have $|x_2| \sim |x_1|^{1/2}$, then from (3.31) we conclude that the left part in (3.31) will be dominated by $|x_1|$, when $x_2 << 1, x_1 << 1$, while the right part will be proportional to $|x_1|^{\frac{5}{6}\frac{1}{\alpha}}$. There exists $\alpha \in (0,1)$ such that $\frac{5}{6}\frac{1}{\alpha} < 1$, hence one can always select $k > 0$ and a neighborhood $N \subset D$ of the origin such that $(\alpha_0\alpha_1)^{\frac{1}{\alpha}}|x_1|^{\frac{5}{6}\frac{1}{\alpha}}$ will be dominated over the right part in (11) which is proportional to $|x_1|$ in a small $N \subset D$. Thus, the proposition (iii) of the theorem is true, and system (3.29) has a finite-time stable equilibrium at the origin for any $\alpha_1 > 0, \alpha_0 > 0$ ■

Thus, if $\beta_{1,1} = 1/2, \beta_{1,0} = 1/3$, system (3.28) converges to the origin in a finite time featuring the second order sliding mode $\sigma_1 = \dot{\sigma}_1 = 0$. In the sliding mode, the compensated tracking error dynamics is governed by Eq.(3.24) under perfect tracking $x_2 \to x_2^c$.

To avoid explosion of terms the virtual control (3.26), i.e.

$$x_2^c = \hat{\dot{y}}_c - \varphi_1(x_1) - \hat{f}_1(.,t) + c_1|e_1|^{\alpha_1}\operatorname{sgn}(e_1) + \rho_0\int|\sigma_1|^{\beta_{1,0}}\operatorname{sgn}(\sigma_1)dt ,$$

must be continuously differentiable twice. This gives us design condition on α_1

$$\tfrac{2}{3} < \alpha_1 < 1,$$

and the following condition must be checked for $\beta_{1,0}$. For the second derivative of x_2^c be bounded, i.e. since $\ddot{x}_2^c \sim \dfrac{\dot{\sigma}_1}{|\sigma_1|^{1-\beta_{1,0}}}$, we must have $\left|\dfrac{\dot{\sigma}_1}{|\sigma_1|^{1-\beta_{1,0}}}\right| < \infty$. From (3.28) we obtain that

$$\dot{\sigma}_1 \sim \sigma_1^{1-\beta_{1,1}} \sigma_1^{\beta_{1,0}}, \text{ then } \ddot{x}_2^c \sim \frac{\dot{\sigma}_1}{|\sigma_1|^{1-\beta_{1,0}}} \sim \frac{\sigma_1^{1-\beta_{1,1}} \sigma_1^{\beta_{1,0}}}{|\sigma_1|^{1-\beta_{1,0}}} \sim \sigma_1^{2\beta_{1,0}-\beta_{1,1}}.$$

So, to avoid explosion of terms one should have $2\beta_{1,0} - \beta_{1,1} \geq 0$. If $\beta_{1,1} = 1/2, \beta_{1,0} = 1/3$, this condition is satisfied.

The second loop design

Dynamics of the tracking error $e_2 = x_2^c - x_2$ is to be enforced in the second loop on the dynamic sliding surface

$$\sigma_2 + \rho_1 \int |\sigma_2|^{\beta_{2,1}} \operatorname{sgn}(\sigma_2) dt = e_2 + c_2 \int |e_2|^{\alpha_2} \operatorname{sgn}(e_2) dt, \tag{3.32}$$

which is similar to the first one (3.23) with one difference, it must be faster enough to enforce sufficient time scale to consider $e_2 = 0$ in the outer loop.

To enforce convergence to $\sigma_2 = 0$, we consider σ_2-dynamics using Eq.(3.1) and introducing the virtual control x_3^c, and the tracking error $e_3 = x_3^c - x_3$ of the inner loop

$$\dot{\sigma}_2 + \rho_{2,1} |\sigma_2|^{\beta_{2,1}} \operatorname{sgn}(\sigma_2) = \dot{x}_2^c - \varphi_2(x_1, x_2) - f_2(.,t) - (x_3^c - e_3) + c_2 |e_2|^{\alpha_2} \operatorname{sgn}(e_2). \tag{3.33}$$

The virtual control x_3^c is designed as

$$x_3^c = \hat{\dot{x}}_2^c - \varphi_2(x_1, x_2) - \hat{f}_2(.,t) + c_2 |e_2|^{\alpha_2} \operatorname{sgn}(e_2) + \rho_{2,0} \int |\sigma_2|^{\beta_{2,0}} \operatorname{sgn}(\sigma_2) dt, \tag{3.34}$$

Given (3.33) and estimation error $\hat{e}_2 = (\dot{x}_2^c - f_2(.,t)) - (\hat{\dot{x}}_2^c - \hat{f}_2(.,t))$, the second loop σ_2-dynamics closed under control (3.34) is obtained

$$\dot{\sigma}_2 + \rho_{2,1} |\sigma_2|^{\beta_{2,1}} \operatorname{sgn}(\sigma_2) + \rho_{2,0} \int |\sigma_2|^{\beta_{2,0}} \operatorname{sgn}(\sigma_2) dt = \hat{e}_2 + e_3. \tag{3.35}$$

Assuming exact estimation, i.e., $\hat{e}_2 = 0$, and fast convergence of e_3 to zero (time scale of the inner loop dynamics), we have the second order finite time convergent σ_2-dynamics.

The third loop design

The very inner loop is to be designed for actual control u to stabilize the tracking error e_3-dynamics

$$\dot{e}_3 = \dot{x}_3^c - \varphi_3(x_1, x_2, x_3) - f_3(.,t) - u. \tag{3.36}$$

The sliding mode control is designed in a standard way [8]

$$u = \varphi_3(x_1, x_2, x_3) + \rho \operatorname{sgn}(e_3), \quad \rho > \left|\dot{x}_3^c\right| + \left|f_3\right|. \tag{3.37}$$

Additional requirements must exist for α_2 in order for $\dot{x}_3^c$ to be bounded. From the obtained condition on smoothness we have $\frac{1}{2} < \alpha_2 < 1$.

The closed loop dynamics in the third loop,

$$\dot{e}_3 = \dot{x}_3^c - f_3(.,t) - \rho \operatorname{sgn}(e_3). \tag{3.38}$$

is a finite time convergent system.

3.4.4 Conclusions on Multiple Time Scale Dynamic SMC Design

A three-loop tracking control system has been designed for a third order uncertain SISO system. Tracking error dynamics is enforced in each loop in terminal dynamic sliding surfaces. The second order sliding performance is provided for sliding modes on dynamic sliding surfaces in the outer and the inner loops under virtual controls. Time scale separation ensures overall stability of Eqs.(3.27).(3.35),(3.38). When $\hat{e}_1 = \hat{e}_2 = 0$, exact tracking $y = y_c$ is achieved in a finite time by system (3.1) under control (3.37).

A signal differentiator and a disturbance observer must accompany the presented design to ensure $\hat{e}_1 \to 0, \hat{e}_2 \to 0$ as close as possible. The design of a signal differentiator and disturbance observer on its basis is presented in the next section on the example of calculating $\hat{\dot{y}}_c$ and $\hat{f}_1(.,t)$.

4. An Auxiliary Tool: Robust SMC Differentiator

2-Sliding Exact Differentiator

The 2-sliding exact differentitator developed in the work [14] can be employed to produce the time derivatives of a smooth reference signal. An exact differentiator of a given signal, $x(t)$, has the form

$$\begin{cases} \hat{x}_1 = \int \hat{x}_2 d\tau, \\ \hat{x}_2 = \rho_1 \dfrac{(x(t) - \hat{x}_1)}{|x(t) - \hat{x}_1|^{0.5}} + \rho_0 \int \mathrm{sgn}(x(t) - \hat{x}_1) d\tau. \end{cases} \tag{4.1}$$

where the output $\hat{x}_2$ converges to $\dot{x}(t)$ in a finite time. To verify this statement, we can identify the dynamics of estimation error, $e = x - \hat{x}$, using Eq. (16) as

$$\ddot{e} + \frac{\lambda}{2} \frac{\dot{e}}{|e|^{0.5}} + \alpha \,\mathrm{sgn}(e) = \ddot{x}(t). \tag{4.2}$$

This is a finite-time-convergence second-order differential equation, which provides for 2-sliding mode, $e = \dot{e} = 0$, in case when the parameters α, λ are appropriately selected in accordance with the upper boundary for $|\ddot{x}(t)| \le L$. A proof to this statement gives the following lemma, which summarizes the results of the work [14] about 2-sliding modes with finite time convergence.

Lemma 2 *Given 2^{nd} order nonlinear differential equation* (4.2) *and conditions* $|\ddot{x}(t)| \le L$, $\alpha \ge 4L$, $\lambda \ge 0.5L^{0.5}$, $e(0) = 0$, $\dot{e}(0) = z_o'$, $z_o' =$ *arbitrary constant, any its solution converges to the origin in finite time.*

Remark The observer (4.1) is subjected to the chattering of the output for noisy signals. Therefore, in practical observer of this form, a low-pass filtering of $\hat{x}_2$ is required for a smooth estimate of $\dot{x}(t)$, as shown in the work [14].

2-Sliding Disturbance Observer

We consider disturbance estimation on the example of the first equation in (3.1), i.e., $\hat{f}_1(.,t)$. The state observer for the system

$$\dot{x}_1 = \varphi_1(x_1) + x_2 + f_1(.), \tag{4.3}$$

is designed as

$$\dot{\hat{x}}_1 = \varphi_1(\hat{x}_1) + x_2 + \rho_1|e|^{0.5}\,\mathrm{sgn}(e) + \rho_0 \int \mathrm{sgn}(e)dt, \tag{4.4}$$

where $e = x_1 - \hat{x}_1$. The observer error dynamics is derived

$$\dot{e} = \varphi_1(x_1) - \varphi_1(\hat{x}_1) + f_1(.) - u_o,\ u_o = \rho_1|e|^{0.5}\,\mathrm{sgn}(e) + \rho_0 \int \mathrm{sgn}(e)dt. \tag{4.5}$$

This is a finite time convergent dynamics given the restriction $\|\varphi_1(x_1) - \varphi_1(\hat{x}_1)\| \le C\|e\|$ for all times. Thus, u_o is an estimate of f_1. To provide smooth derivative, the estimate u_o is low-pass filtered before entering the virtual control as $\hat{f}_1$

$$\hat{f}_1 = \frac{1}{(Ts+1)^k} u_o. \tag{4.6}$$

5. Simulation Example

To illustrate the disturbance cancellation characteristics of the developed method, we consider the simplified model of a ballistic interceptor missile. A simplified model in the pitch plane is given by

$$\begin{cases} \dot{v}_x = -\dfrac{1}{m}(u_a + u_d)\sin\theta + \dfrac{\psi_x}{m} \\ \dot{v}_z = \dfrac{1}{m}(u_a + u_d)\cos\theta + \dfrac{\psi_z}{m} - g \end{cases}, \tag{5.1}$$

where $V = \sqrt{v_x^2 + v_z^2}$ is the velocity vector, m is the mass of the interceptor in kg, ψ_x and ψ_z are disturbances, g=9.81m/s^2 is gravity, θ is the pitch angle of the interceptor, u_a is the attitude control and u_d is the divert control.

A simplified rotational motion model for the interceptor is given by

$$\begin{cases} \dot{\theta} = q, \\ \dot{q} = -\dfrac{r}{J}u_a + \dfrac{\psi_\omega}{J}, \end{cases} \tag{5.2}$$

where q is the pitch rate (*rad/s*), J is the moment of inertia of the interceptor (kg·m^2), r is the distance between application of the attitude control force u_a at the interceptor's center of gravity, and ψ_ω is the rotational disturbance term. The output equations are given by

$$\begin{cases} y_1 = A_n = \dot{\gamma} \cdot V \\ y_2 = \alpha = \theta - \gamma \end{cases}, \tag{5.3}$$

where γ is the missile's flight path angle, α is the angle of attack, and A_n is the normal acceleration. Our goal is to design the attitude control thrust to stabilize angle of attack to zero, assuming that the divert control thrust delivers desired normal acceleration.

$$\begin{aligned} \lim_{t\to\infty}\left|y_{1r}(t) - y_1(t)\right| = \lim_{t\to\infty}\left|A_{nr} - A_n\right| = 0 \\ \lim_{t\to\infty}\left|y_{2r}(t) - y_2(t)\right| = \lim_{t\to\infty}\left|0 - \alpha\right| = 0 \end{aligned}, \tag{5.4}$$

where A_{nr} is a desired reference profile for normalized acceleration. The reference profile for the normalized acceleration is generated by the guidance system to minimize miss distance in the interceptor using a number of methods, such as proportional navigation, augmented proportional navigation and other advanced guidance laws [1]. Forcing α to zero aligns the missile body with

the flight path angle. We consider the normal aerodynamic lift force and pitching moment as disturbances entering the plant

$$N = \frac{\rho_a V^2}{2} S \cdot C_{N_\alpha} \cdot \alpha\,,\;\; M = \frac{\rho_a V^2}{2} Sl \cdot C_{m_\alpha} \cdot \alpha\,,$$

$$\psi_x = -N \sin\gamma\,,\; \psi_z = N\cos\gamma\,,\; \psi_a = M\,.$$

Finally our numerical model is obtained as

$$\begin{aligned}
\dot{v}_x &= -\frac{1}{70}\left((u_a + 1000)\sin\theta + \psi_x\right),\\
\dot{v}_z &= \frac{1}{70}\left((u_a + 1000)\cos\theta + \psi_z\right) - 9.81,\\
\dot{\theta} &= q,\\
\dot{q} &= -0.1(u_a + \psi_a),\\
\alpha &= \theta - \gamma,\\
\gamma &= \tan^{-1}\left(\frac{v_z}{v_x}\right),\\
V &= \sqrt{v_x^2 + v_z^2},\\
\psi_x &= -3\cdot 10^{-4} V^2 \alpha \sin\gamma,\; \psi_z = 3\cdot 10^{-4} V^2 \alpha\cos\gamma,\; \psi_a = -0.5\cdot 10^{-6} V^2\alpha,\\
v_x(0) &= 2000 m/s, v_z(0) = 0 m/s, \theta(0) = 0.3 rad,\; q(0) = 0 rad/s.
\end{aligned} \tag{5.5}$$

The plant output to be stabilized is angle of attack, the attitude control trust is actuated via a first order actuator

$$\dot{u}_a = -20(u_a - u)\,, \tag{5.6}$$

so the actual control signal is u, and we have a third order input-output dynamics

$$\begin{aligned}
\dot{\alpha} &= q - \frac{A_n}{V},\\
\dot{q} &= -0.1(u_a + \psi_a),\\
\dot{u}_a &= -20(u_a - u),
\end{aligned} \tag{5.7}$$

where we consider $A_n = -\dot{v}_x \sin\gamma + \dot{v}_z \cos\gamma$ and ψ_a as disturbances in the first and second loops of a 3-loop control system to be designed.

The stabilizing controller is designed as follows. The first dynamic sliding surface is selected as

$$\sigma_1 + 0.5\int |\sigma_1|^{1/2}\,\mathrm{sgn}(\sigma_1)dt = \alpha + 0.05\int |\alpha|^{2/3}\,\mathrm{sgn}(\alpha)dt\,. \tag{5.8}$$

Then, σ_1-dynamics are identified, introducing $e_q = q_c - q$,

$$\dot{\sigma}_1 = -0.5|\sigma_1|^{1/2}\,\mathrm{sgn}(\sigma_1) + 0.05|\alpha|^{2/3}\,\mathrm{sgn}(\alpha) + q_c - e_q - \frac{A_n}{V}.$$

Virtual control in the first loop is designed

$$q_c = -0.05|\alpha|^{2/3}\,\mathrm{sgn}(\alpha) - 1.5\int |\sigma_1|^{1/3}\,\mathrm{sgn}(\sigma_1)dt\,. \tag{5.9}$$

The second dynamic sliding surface with appropriate time scale is selected as

$$\sigma_2 + 3\int |\sigma_2|^{1/2}\,\mathrm{sgn}(\sigma_2)dt = e_q + 0.5\int |e_q|^{1/2}\,\mathrm{sgn}(e_q)dt\,. \tag{5.10}$$

Then, σ_2-dynamics are identified, introducing $e_u = u_{ac} - u_a$,

$$\dot{\sigma}_2 = -3|\sigma_2|^{1/2}\,\mathrm{sgn}(\sigma_2) + 0.5|e_q|^{1/2}\,\mathrm{sgn}(e_q) + \dot{q}_c + 0.1(u_{ac} - e_u) - 0.1\psi_a.$$

Virtual control in the second loop is designed

$$u_{ac} = 10\cdot\left(-\dot{q}_c - 0.5|e_q|^{1/2}\,\mathrm{sgn}(e_q) - 10\int |\sigma_2|^{1/3}\,\mathrm{sgn}(\sigma_2)dt\right). \tag{5.11}$$

Tracking error dynamics in the third loop are identified

$$\dot{e}_u = \dot{u}_{ac} + 20u_a - 20u\,.$$

Actual discontinuous control is designed

$$u = 20\,\mathrm{sgn}(e_u)\,. \tag{5.12}$$

Finite time convergent multiple sliding surfaces dynamics are obtained

$$\begin{aligned}
&\dot{\sigma}_1 + 0.5|\sigma_1|^{1/2}\,\mathrm{sgn}(\sigma_1) + 1.5\int |\sigma_1|^{1/3}\,\mathrm{sgn}(\sigma_1)dt = -e_q - \frac{A_n}{V},\\
&\dot{\sigma}_2 + 3|\sigma_2|^{1/2}\,\mathrm{sgn}(\sigma_2) + 10\int |\sigma_2|^{1/3}\,\mathrm{sgn}(\sigma_2)dt = -e_u - 0.1\psi_a,\\
&\dot{e}_u = \dot{u}_{ac} + 20u_a - 400\,\mathrm{sgn}(e_u).
\end{aligned} \tag{5.13}$$

Analyzing (5.13), one can conclude that e_u reaches zero in a finite time robustly to bounded behavior of $\dot{u}_{ac}$, u_a. When $e_u = 0$, σ_2 approaches in a finite time a small domain around zero attenuating the disturbance ψ_a. When $\sigma_2 \approx 0$, e_q goes to zero in a finite time according to $\dot{e}_q = -0.5|e_q|^{0.5}\,\mathrm{sgn}(e_q)$. When $e_q \approx 0$, σ_1 approaches in a finite time a small domain around

zero attenuating the disturbance $\frac{A_n}{V}$. When $\sigma_1 \approx 0$, α goes to zero in a finite time according to $\dot{\alpha} = -0.05|\alpha|^{2/3}\,\mathrm{sgn}(\alpha)$. Results of a simulation are given in Figs.

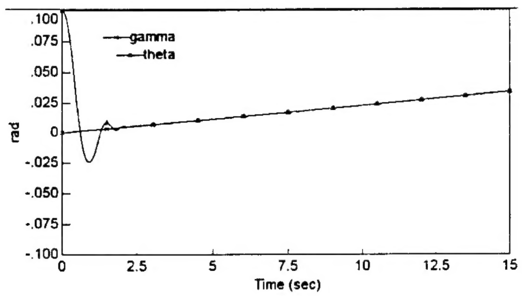

Fig.4 γ, θ versus time

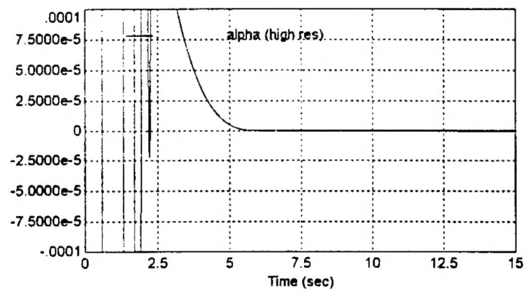

Fig.5 angle of attack vs.time

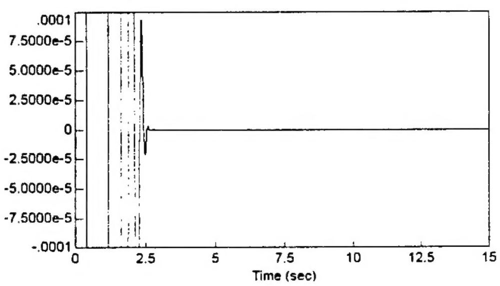

Fig.6 σ_1 versus time

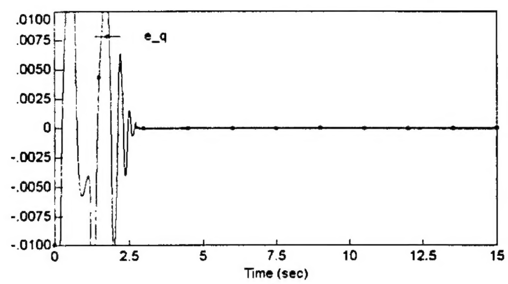

Fig.7 pitch rate tracking error vs.time

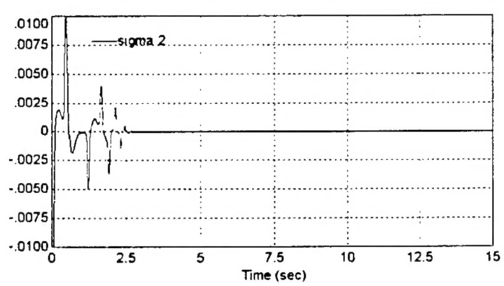

Fig.8 σ_2 versus time

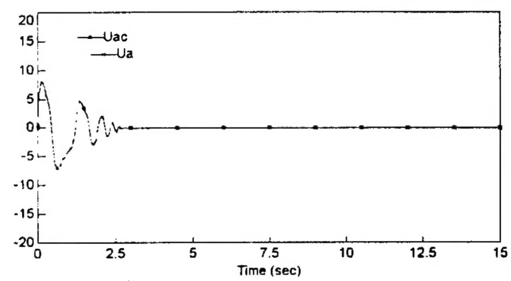

Fig.9 Tracking attitude thrust command

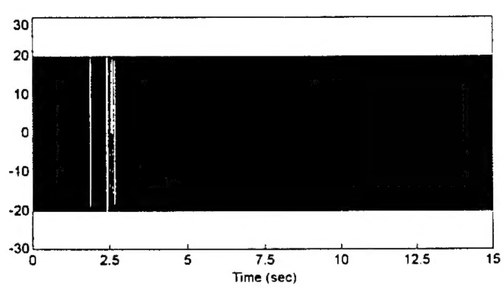

Fig.10 Control u versus time

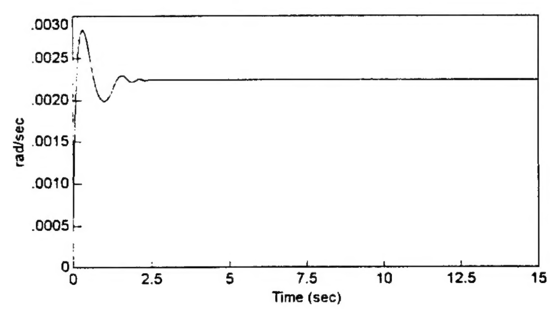

Fig.11 Disturbance $\frac{A_n}{V}$

6. Scientific Merit and Impact of the Research

- The new smooth sliding mode controllers that employ first order and second order sliding modes and sliding mode observers are proposed to stabilize multiple sliding surfaces in the sliding mode controlled plant.
- The proposed smooth sliding mode controllers can be used is control systems where control actions must be smooth, in particular, in multiple loop control systems.
- Under the assumption that the observer captures the uncertainty and cancels its effect on the closed-loop dynamics, the system motion in the vicinity of the sliding surface is governed by a second-order finite time convergent dynamics that establishes a second-order sliding on the sliding surface. Thus, both parts of the control law (feedback PI-type control and disturbance observer) are the second order SMC's. Under the discrete-time control with a zero-order hold, the accuracy of holding the trajectories on the sliding surface is of the second-order real sliding, $O(T^2)$.

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